A Geometrical Application of the Theory of the Binary Quintic.

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Introduction.

If five points on a non-degenerate conic are given, certain curves arise from the polarization of the binary quintic determined by them. These are covariant curves of the five points. It is the object of this paper to discuss certain of these curves, and sets of points and lines associated with them. Similar processes may be applied to the quintic of five points on the cubic norm curve in space. A brief treatment of the covariant forms of the binary quintic from this standpoint is given in § 7, and the results are correlated as far as possible with those already obtained in the plane.

For the algebraic theory of the quintic, reference is made to Salmon's "Higher Algebra" and to "Algebra of Invariants" by Grace and Young. The notations of both works are used.

§ 1. *A Pencil of Conics.*

Let the quintic be represented on the base conic N, and let the reference triangle be formed by the tangents to N at the points given by the canonizant $C_{3,3}=0$. The quintic may then be written in Salmon's canonical form,

$$f\equiv a_0t^5+a_1(1-t)^5-a_2, \tag{1}$$

and the conic in the form

$$x_0=t^2, \qquad x_1=(1-t)^2, \qquad x_2=1. \tag{2}$$

Any point of the plane, in terms of the parameters of points of contact of tangents from it to N, is

$$x_0=t_1t_2, \qquad x_1=(1-t_1)(1-t_2), \qquad x_2=1; \tag{3}$$

and any line of the plane, in terms of its intersections with N, is

$$\xi_0=t_1+t_2-2, \qquad \xi_1=-(t_1+t_2), \qquad \xi_2=t_1+t_2-2t_1t_2. \tag{4}$$

The point $(1, 1, 1)$ is given by the Hessian of the canonizant, the $C_{6,2}$ $\equiv a_0^2a_1^2a_2^2\{t^2+(1-t)^2+1\}$ of f. Every quadratic determines at once a point and a line of the plane, these being pole and polar with respect to N.

If the quintic be polarized thus:

$$a_0t_1t_2^2t_3^2+a_1(1-t_1)(1-t_2)^2(1-t_3)^2-a_2=0, \tag{5}$$

the locus of the intersection of tangents at t_2 and t_3 for a fixed t_1 is a conic whose equation is

$$a_0t_1x_0^2+a_1(1-t_1)x_1^2-a_2x_2^2=0. \tag{6}$$

By varying t_1 we obtain a pencil of conics apolar to N (in lines) and on the four points $\pm\sqrt{a_0^{-1}}$, $\pm\sqrt{a_1^{-1}}$, $\pm\sqrt{a_2^{-1}}$. These four points are a set orthic to N. When $t_2=t_3$, the conic t_1 meets N, and the points of intersection are given by the quartic polar of t_1 as to f. t_1 is a root of $f=0$ when, and only when, point t_1 of N is on conic t_1 of the pencil. If t_1 is a root of the canonizant (*i. e.*, $t_1=0, 1, \infty$), the corresponding conic degenerates into lines meeting at a reference point; hence, the reference triangle ABC (Fig. 1) is the diagonal triangle

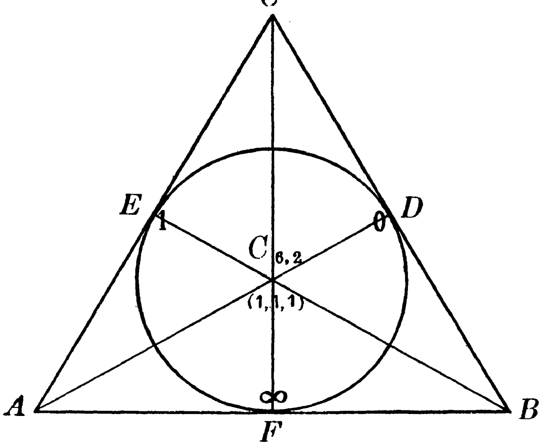

FIG. 1.

of the base-points of the pencil. Since the roots of the Hessian of f are double points of quartic polars, there are six members of the pencil tangent to N, and their points of contact are the points $H=0$. No member has double contact unless an invariant of f vanishes. Two values of t_1 (given by $C_{2,2}=0$) have quartic polars which are self-apolar; the corresponding conics are apolar to N both in points and in lines.*

Conics of the pencil and points of N are in one-to-one correspondence. We can construct geometrically the point corresponding to a given conic, and *vice versa*. The point (1, 1, 1) is given by the quadratic $C_{6,2}=0$; its polar lines as to all conics of the pencil pass through the point $Q(a_0^{-1}, a_1^{-1}, a_2^{-1})$ given by the quadratic $C_{2,2}\equiv a_1a_2(1-t)+a_0a_2t-a_0a_1t(1-t)=0$. The polar of Q as to any member of the pencil therefore passes through (1, 1, 1), and its coordinates are

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* W. F. Meyer, "Apolarität und rationale Curven," p. 150.

$$\eta_0=t_1, \qquad \eta_1=1-t_1, \qquad \eta_2=1. \tag{7}$$

If the conic t_1 is given, we construct this line η and take its polar point with respect to the triangle DEF whose vertices are the canonizant points on N. This point is t_1. The proof follows at once from consideration of the degenerate members. For, it has been seen that the t_1's of the degenerate conics are 0, 1, ∞, which give the points D, E, F on N; and the polar of Q as to the degenerate conic on A is the line $A-(1, 1, 1)-D$, whose polar point as to the triangle DEF is D. By the fundamental theorem on projective correspondences, since the construction holds for three members of the pencil it holds for all. It is obviously reversible. The points $C_{2,2}=0$ and $C_{6,2}=0$ are mutually related to the pencil.

The above correspondence of point and conic is easily verified analytically.

§ 2. *Rational Quartic K of Class* 3.

Let f be polarized thus:

$$a_0t_1^3t^2+a_1(1-t_1)^3(1-t)^2-a_2=0, \tag{8}$$

and consider the locus of the line joining the roots of the quadratic in t. We obtain a rational line cubic K whose equations are

$$\xi_0=a_0t_1^3, \qquad \xi_1=a_1(1-t_1)^3, \qquad \xi_2=-a_2. \tag{9}$$

ξ passes through a given point $x_0=t_2t_3$, $x_1=(1-t_2)(1-t_3)$, $x_2=1$, provided t_1 satisfies

$$a_0t^3t_2t_3+a_1(1-t)^3(1-t_2)(1-t_3)-a_2=0. \tag{10}$$

This is the cubic polar of t_2t_3 as to f, and cubic polars are all cubics apolar to the canonizant. Hence, the necessary and sufficient condition that three t's be parameters of lines on a point is that they be apolar to $C_{3,3}$. The curve has three cusps; for, if t_2, t_3 be taken as two roots of $C_{3,3}=0$, the cubic polar becomes a perfect cube, the cube of the third root. Hence the points A, B, C are the cusps. Since $C_{3,3}$ is the cubic polar of $C_{2,2}$, the cusp tangents meet at $C_{2,2}=0$ or Q of § 1. For a given t_1 the line ξ is identical with the polar of point t_1 on N as to the corresponding member of the pencil (6).

By differentiating the functions ξ of (9) we obtain the equations of K as a point quartic:

$$x_0=a_1a_2(1-t)^2, \qquad x_1=a_0a_2t^2, \qquad x_2=a_0a_1t^2(1-t)^2. \tag{11}$$

The line equations of N are

$$\xi_0=(1-t), \qquad \xi_1=t, \qquad \xi_2=-t(1-t). \tag{12}$$

Hence, the quadratic of points of contact of tangents from a given point t_1 of K to N is:

$$a_1a_2(1-t_1)^2(1-t)+a_0a_2t_1^2t-a_0a_1t_1^2(1-t_1)^2t(1-t)=0. \tag{13}$$

This is the Hessian of the cubic polar of t_1 as to f; namely,

$$a_0t_1^2t^3+a_1(1-t_1)^2(1-t)^3-a_2.$$

The curve K may therefore be regarded as produced from N by a transformation T which sends the point t of N into the intersection of tangants at the Hessian of the cubic polar of t as to f. Hence, to find the parameters of the intersections of K and N, we ask for a point t_1 such that the Hessian of its cubic polar is a perfect square; that is, we look for t_1t_2 such that $t_1^2t_2^2$ is a quartic apolar to f. If f be written symbolically, this covariant is found by elimination of t_1 between $(\alpha t_1)^3(\alpha t)^2=0$ and $(\alpha t_1)^2(\alpha t)^3=0$. f factors out of the eliminant, and the remaining factor is a $C_{4,8}$. Now the Hessian of the cubic polar $(\alpha t_1)^2(\alpha t)^3$ is $|\alpha\beta|^2(\alpha t_1)^2(\beta t_1)^2(\alpha t)(\beta t)$. Equated to zero, this function gives a convenient representation of the curve, t_1 being the parameter and a point of the curve being given by the two values of t. K cuts N when the two t's are equal. This gives

$$C_{4,8}\equiv|\alpha\beta|^2|\gamma\delta|^2|\alpha\gamma|\,|\beta\delta|\,(\alpha t)^2(\beta t)^2(\gamma t)^2(\delta t)^2=0, \tag{14}$$

where all of the symbols refer to the quintic. The roots of $C_{4,8}$ occur in four pairs, and the square of each pair is a quartic apolar to f. Taking t_2 and t_3 of (5) as a pair, it is evident that the points of the plane determined by these four quadratics are the base-points of the pencil (6).

The common lines of K and N have as parameters the six Hessian points of f, and their points of contact with N are the roots of the Steinerian. This follows at once from the fact that the lines of K are cut out by quadratic polars of points t as to f. To find the double line of K, we recall that parameters of lines on a point form cubics apolar to $C_{3,3}$; hence, the parameters of the double line are given by the Hessian of $C_{3,3}$; *i. e.*, $C_{6,2}\equiv a_0^2a_1^2a_2^2(t^2-t+1)=0$. A root of this is $-\omega$, and the corresponding line of K is the line (a_0, a_1, a_2). Hence, *the double line of K is the polar line of $C_{2,2}$ as to the triangle ABC.*

K in x coordinates is

$$\sum^3 a_1^2a_2^2x_1^2x_2^2-2\sum^3 a_0^2a_1a_2x_0^2x_1x_2=0. \tag{15}$$

§ 3. *The Involutory Quadratic Transformation T.*

Mention was made in the preceding section of a point transformation T which transforms N into the quartic K. This transformation may be arrived at in three ways. It is:

(a) The transformation which sends any point x of the plane into the point y given by the Hessian of the cubic polar (as to f) of the quadratic of x.

(b) The transformation which sends x into a point y such that the quadratics of x and y from a quartic apolar to f.

(c) The transformation which sends x into the intersection of its polar lines as to the pencil (6) of § 1.

That (a) and (b) are equivalent follows from the fact that a quadratic and the Hessian of its cubic polar as to f obviously form a quartic apolar to f, and that when two roots of an apolar quartic (which are not roots of $C_{3,3}=0$) are given, the remaining two are uniquely determined. The matter may be stated thus: If, of two quadratics Q_1 and Q_2, Q_2 is the Hessian of the cubic polar of Q_1 as to a quintic f, then Q_1 is a factor of the Hessian of the cubic polar of Q_2. As an instance, we note that the cubic polar of $C_{2,2}$ is $C_{3,3}$, whose Hessian is $C_{6,2}$. The cubic polar of $C_{6,2}$ is $C_{7,3}$, whose Hessian must therefore contain $C_{2,2}$ as a factor.

To show that (a) and (c) are equivalent, we derive a formula for (c). The conic (6), when polarized, gives

$$a_0t_1x_0y_0+a_1(1-t_1)x_1y_1-a_2x_2y_2=0. \tag{16}$$

If y be given, this line for all t_1's passes through $(a_iy_i)^{-1}$, $i=0, 1, 2$. Hence, (c) is the transformation

$$x_i=(a_iy_i)^{-1}, \qquad i=0, 1, 2. \tag{17}$$

Now let t_1t_2 be roots of the quadratic of x, and form the cubic polar $a_0t_1t_2t^3+a_1(1-t_1)(1-t_2)(1-t)^3-a_2$ and its Hessian $a_1a_2(1-t_1)(1-t_2)(1-t)+a_0a_2t_1t_2t-a_0a_1t_1t_2t(1-t_1)(1-t_2)(1-t)$. If the roots of this quadratic in t are t_3 and t_4, we find as the corresponding point of the plane

$$\begin{aligned}y_0&=t_3t_4=a_1a_2(1-t_1)(1-t_2),\\ y_1&=(1-t_3)(1-t_4)=a_0a_2t_1t_2,\\ y_2&=a_0a_1t_1t_2(1-t_1)(1-t_2).\end{aligned}$$

Using (3), this reduces to $y_0=a_1a_2x_1x_2$, $y_1=a_0a_2x_0x_2$, $y_2=a_0a_1x_0x_1$, which is identical with the form (17).

From (c), the fixed points of T are the base-points of pencil (6). From (b), a fixed point determines a quadratic t_1t_2 such that $t_1^2t_2^2$ is apolar to f. It was shown in § 2 that t_1, t_2 are then intersections of K with N; that is, K cuts N at the points of contact of tangents to N from the fixed points of T. That A, B, C are the singular points of T is obvious from the fact that $C_{3,3}$ (whose roots taken in pairs give A, B, C) is apolar to f. For, if t_1, t_2, t_3 are roots of $C_{3,3}=0$, $t_1t_2t_3t$ is apolar to f for an arbitrary t; that is, the point A transforms into any point of the line BC. From (c), it may be seen that corresponding points x, y as to T are base-points of the involution cut on the line $\overline{xy}$ by the conics (6).

§ 4. *Rational Quintic O and a Theorem Concerning its Double Points.*

If we polarize f as follows:

$$a_0t_1^4t+a_1(1-t_1)^4(1-t)-a_2=0, \tag{18}$$

and consider the locus of the intersection of tangents to N at t_1 and t, we obtain a rational curve with parameter t_1 of order 5 and class 8. This quintic O is the locus of the intersection of the tangent to N at t_1 with the corresponding line of K; for, the t_1 and t which satisfy $(\alpha t_1)^4(\alpha t)=0$, (18), are apolar to the quadratic $(\alpha t_1)^3(\alpha t)^2$ (8) which determines the line t_1 of K. When t_1 is a root of $f=0$, the equations $(\alpha t_1)^4(\alpha t)=0$ and $(\alpha t_1)(\alpha t)^4=0$ are both satisfied by t_1; hence, two points of O on the tangent to N at t_1 have come into coincidence at t_1, and *O is therefore five times tangent to N at the points $f=0$.* This accounts for ten intersections and ten common lines. To find the remaining six common lines, let h and s be a pair of corresponding roots of Hessian and Steinerian of f, and let the tangents at these points on N meet at R. Since the quartic polar of s as to f has a double root at h, Rs is tangent to O at R, and the six lines are accounted for. Moreover, since h^3s is a quartic apolar to f, the point h is transformed into R by T (using § 3,(b)), and R is a point of K. It has been seen (§ 2) that Rs is the tangent to K at R. *O and K are therefore six times tangent at the points R, and their common tangents at these points are tangent to N.*

O has six double points with parameters which are pairs of solutions of $(\alpha t_1)^4(\alpha t)=0$ and $(\alpha t_1)(\alpha t)^4=0$. Elimination of t_1 gives a $C_{5,17}$ which obviously contains f as a factor. The remaining factor $C_{4,12}$ is the product of the six quadratics which determine the points. Since the number of constants used to determine them is not twelve but ten, this set of six points are subject to two invariant conditions. Concerning them we will prove the following theorem:

If P is a double point of O, there exists a point γ on N and on line PQ, such that the conic C_γ of (6) *is tangent to PQ at P, and the line η_γ* (7) *passes through P. γ is the polar of the two parameters of P as to $C_{3,3}$ and satisfies a certain $C_{6,6}=0$.* (Q is point $C_{2,2}=0$, as in § 1.) (19)

To prove this, let t_1, t_2 be the parameters of P, and let γ be determined as the polar of t_1t_2 with respect to $C_{3,3}$. Since t_1 and t_2 are a pair of solutions of $(\alpha t_1)^4(\alpha t_2)=0$ and $(\alpha t_2)^4(\alpha t_1)=0$, the cubic polar of t_1t_2 as to f is $t_1t_2\gamma$; that is, the lines of K having parameters t_1, t_2, γ pass through P. Since the cubic $t_1t_2\gamma$ is apolar to itself, we have $t_1t_2t_1t_2\gamma$ apolar to f, or $(\alpha t_1)^2(\alpha t_2)^2(\alpha t)=0$ when $t=\gamma$. But this is the condition that the conic C_γ should pass through P. From the fact that the cubic polar $t_1t_2\gamma$ is apolar to $C_{3,3}$, it follows that η_γ (which is the polar line of γ as to triangle DEF) passes through P. Now η_γ is the polar of Q as to conic C_γ; and line K_γ (which has been shown to pass

through P) is the polar of γ as to C_γ (§ 2). Hence, the points Q and γ are on the tangent to C_γ at P, which establishes the theorem.*

The six parameters γ are roots of a covariant $C_{6,6} \equiv C_{5,1}C_{1,5} - C_{3,3}^2$. This follows immediately from the theorem, as will be shown in the following section.

§ 5. *Curves on the Double Points of O: Cubic Ω and Rational Cubic Ψ.*

The theorem (19) of the preceding section gives certain curves on the double points P of O and having interesting relations to the system of covariants of f.

Since the points P are points of contact of tangents from $C_{2,2}$ to members of pencil (6), consider the locus of such points. This is a non-rational cubic curve Ω on the four base-points of the pencil and tangent at these points to the lines joining them to $C_{2,2}$, which latter point is on the curve.† Ω is apolar to N and cuts it in $C_{4,6} \equiv [f, C_{3,3}]^1$. Its equation is

$$\overset{3}{\Sigma} a_0 x_0^2 (x_1 - x_2) = 0. \tag{20}$$

The curve passes through A, B, C; the tangents at these points and the tangent at $C_{2,2}$ meet at $C_{6,2}$ on the curve. On any line through $C_{2,2}$ there are two points of the curve which by the transformation T are interchanged (§ 3, (c)); hence, Ω is by T transformed into itself. $C_{2,2}$ and $C_{6,2}$ are symmetrically related to the pencil. Interchanging their rôles, we have a second cubic on $A, B, C, C_{6,2}, C_{2,2}$ and the four base-points; these nine points are therefore the base-points of a pencil of cubics each of which transforms into itself by T.

For convenience, the coordinates of certain lines which are readily expressed in terms of the parameter t of a point S on N are given here:

$$\left.\begin{array}{llll}
(1)\ \text{Tangent to } N: & 1-t, & t, & -t(1-t). \\
(2)\ \text{Line } SC_{2,2}: & \dfrac{(1-t)^2}{a_2} - \dfrac{1}{a_1}, & \dfrac{1}{a_0} - \dfrac{t^2}{a_2}, & \dfrac{t^2}{a_1} - \dfrac{(1-t)^2}{a_0}. \\
(3)\ \text{Line } \eta: & t, & 1-t, & -1. \\
(4)\ \text{Line } t \text{ of } K: & a_0 t^3, & a_1(1-t)^3, & -a_2. \\
(5)\ \text{Line } \zeta: & a_0 t, & a_1(1-t), & -a_2.
\end{array}\right\} \tag{21}$$

η, it will be recalled, is the polar of S as to triangle DEF, and the polar of $C_{2,2}$ as to conic t of pencil (6); ζ is the polar of $C_{6,2}$ as to the same conic. The

* In the above discussion we considered a pencil of conics correlated to points of a conic N. Joining points of N to Q ($C_{2,2}=0$), we have a (1, 2) correspondence of conics of the pencil and lines on $C_{2,2}$. When a conic touches its line, the point of contact is one of the six points P. When $C_{2,2}$ is replaced by an arbitrary fixed point A of the plane, the six points of contact are (presumably) a general set of six, since the number of their coordinates is now 12. Placing A at $C_{2,2}$ produces the specialization peculiar to the double points of O.

† Salmon-Fiedler, "Geometrie der höheren ebenen Kurven," p. 269.

equation of Ω is obtained by eliminating t between the equation of η and that of conic t of (6).

Any two of the above lines intersect in a point whose locus is a rational curve. The following combinations are noteworthy:

(a) (1) (2) : Conic N.
(b) (1) (4) : Quintic O.
(c) (2) (3) : Cubic Ψ with double point at $C_{6,2}$.
(d) (2) (4) : A quintic with triple point at $C_{2,2}$.
(e) (3) (4) : Quartic Θ with triple point at $C_{6,2}$.
(f) (3) (5) : Conic F on A, B, C, $C_{2,2}$ and $C_{6,2}$.

That (c), (d) and (e) pass through the double points P of O follows at once from the theorem (19). When t is γ of the theorem, the lines (2), (3) and (4) meet at P; hence, the determinant of the coordinates (2), (3) and (4) is the sextic whose roots are the six γ's. This sextic is reducible to $C_{6,6} \equiv C_{5,1}C_{1,5} - C_{3,3}^2$.*

The cubic Ψ is defined by lines (2) and (3), from which its parametric equations are at once obtained. Elimination of t gives

$$\sum^{3} a_0^{-1}\{x_1(x_0-x_1)^2 - x_2(x_0-x_2)^2\} = 0. \tag{22}$$

Since η always passes through $C_{6,2}$, and line (2) passes through this point when t satisfies $[C_{6,2},\ C_{2,2}]^1 = C_{8,2} = 0$, Ψ has a double point at $C_{6,2}$ with the parameters $C_{8,2}=0$. It cut N in $C_{3,3}=0$ (the points D, E, F) and in $C_{5,3} = [C_{2,2},\ C_{3,3}]^1 = 0$. η passes through $C_{2,2}$ when t satisfies $[C_{2,2},\ C_{3,3}]^2 = C_{5,1} = 0$. Hence, Ψ passes through $C_{2,2}$ with $C_{5,1}$ as parameter. Ω and Ψ have three intersections at $C_{6,2}$ and $C_{2,2}$, and their remaining intersections are the six points P. Ω and a given point $C_{2,2}$ on it determine these points, since $C_{6,2}$ and A, B, C are thereby given; varying the position of $C_{2,2}$ on the curve gives ∞^1 such sets.

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*Let f be represented by $C_{1,5} = (at)^5 = (\epsilon t)^5 = (\delta t)^5 = (\zeta t)^5$. Then $C_{2,2} = |a\delta|^4(at)(\delta t)$. Let the canonizant be represented by $(\beta t)^3$. Then $(\beta t)^3 = C_{3,3} = [C_{2,2},\ C_{1,5}]^2 = |a\delta|^4|a\zeta||\delta\zeta|(\zeta t)^3$. $C_{5,1} = [C_{3,3},\ C_{2,2}]^2 = [(\beta t)^3,\ |a\delta|^4(at)(\delta t)]^2 = |a\delta|^4|a\beta||\delta\beta|(\beta t)$. Also let $(\gamma t_1) = \gamma_0 + \gamma_1 t_1 = 0$, γ being real. The three lines η, K and $SC_{2,2}$ cut from N the three quadratics $(\beta t_1)(\beta t)^2 = 0$, $(at_1)^3(at)^2 = 0$ and $|a\delta|^4(at_1)(\delta t)(\gamma t) = 0$ respectively. The three lines meet in a point when the three quadratics are in involution; *i.e.*, when the determinant of the coefficients of t vanishes. The condition to be satisfied by t_1 is

$$\begin{array}{r} (\beta t_1) \\ (at_1)^3 \\ |\zeta\delta|^4(\delta t_1) \end{array} \begin{vmatrix} \beta_0^2 & 2\beta_0\beta_1 & \beta_1^2 \\ a_0^2 & 2a_0a_1 & a_1^2 \\ \zeta_0\gamma_0 & \zeta_0\gamma_1 + \zeta_1\gamma_0 & \zeta_1\gamma_1 \end{vmatrix} = 0.$$

Since $(\gamma t_1) = 0$, $|\beta\gamma| = (\beta t_1)$ and $|a\gamma| = (at_1)$. Expanding the determinant and dropping the subscript from t_1,

$$|\zeta\delta|^4(at)^3(\beta t)(\delta t)|a\beta|\{|\beta\gamma||\zeta a| + |a\gamma||\zeta\beta|\} = 0.$$

Using the identity $|a\beta|(\delta t) \equiv |a\delta|(\beta t) - |\beta\delta|(at)$,

$$|\zeta\delta|^4(at)^3|a\delta||\zeta a|(\beta t)^3 + |\zeta\delta|^4|a\delta||\zeta\beta|(at)^4(\beta t)^2 - |\zeta\delta|^4|\beta\delta||\zeta a|(at)^4(\beta t)^2 - |\zeta\delta|^4|\beta\delta||\zeta\beta|(\beta t)(at)^5 = 0.$$

Since $\zeta \equiv \delta$, the second and third terms cancel. The first and last terms give $C^2_{3,3} - C_{5,1}C_{1,5} = 0$.

The locus of points of contact of tangents from S to the corresponding conic of (6) is a septimic on the points P. Its equation is

$$\overset{3}{\Sigma} a_0 x_0 (a_2 x_2^2 - a_1 x_1^2)^3 = 0,$$

gotten by eliminating t from (6) of § 1 and line (4) of the present section.

§ 6. *Quartic Θ with Triple Point.*

The locus of the intersection of lines (3) and (4) of § 5 (21) is a rational quartic Θ. It has certain projective peculiarities which will be discussed.

The parametric equations of Θ are:

$$\left.\begin{aligned} x_0 &= (1-t)\,[a_1(1-t)^2 - a_2], \\ x_1 &= t\,[a_2 - a_0 t^2], \\ x_2 &= t(1-t)\,[a_1(1-t)^2 - a_0 t^2]. \end{aligned}\right\} \tag{23}$$

Elimination of t gives the form

$$\overset{3}{\Sigma} a_0 x_0 (x_2 - x_1)^3 = 0. \tag{24}$$

To obtain a symbolic expression, we recall that line η cuts N in the quadratic polar of t as to $C_{3,3}$; line K cuts from N the quadratic polar of t as to f (§ 2). If we write $C_{3,3} \equiv (\zeta t)^3$ and $f \equiv (\alpha t)^5 \equiv (\beta t)^5 \equiv (\gamma t)^5 \equiv (\delta t)^5$, the point x of the curve is given by the Jacobian of the quadratics in t', $(\zeta t)(\zeta t')^2$ and $(\delta t)^3 (\delta t')^2$; that is, $|\zeta\delta|\,(\zeta t)\,(\delta t)^3 (\zeta t')\,(\delta t') = 0$. Since $C_{3,3} \equiv |\alpha\beta|^4 |\alpha\gamma|\,|\beta\gamma|\,(\gamma t)^3$, this becomes $|\alpha\beta|^4 |\alpha\gamma|\,|\beta\gamma|\,|\delta\gamma|\,(\delta t)^3 (\gamma t)\,(\delta t')\,(\gamma t') = 0$, where t is the parameter and the two t''s are points on N whose tangents meet at point t of the quartic. η always passes through $C_{6,2}$; line K passes through this point when $[C_{6,2}, f]^2 \equiv C_{7,3} = 0$. Hence, the curve has a triple point at $C_{6,2}$ with parameters $C_{7,3} = 0$. The tangents at the triple point are the lines η having these parameters.

The vertices of the reference triangle ABC are on the curve, and its sides make sections which break into harmonic pairs.* This is seen at once from the equations (23). It is natural to ask for the number of triangles thus related to the curve. The contravariant of the general ternary quartic $(\alpha x)^4$ which, when equated to zero, gives the locus of lines making sections such that $g_3 = 0$, is $|\alpha\beta\xi|^2 |\beta\gamma\xi|^2 |\gamma\alpha\xi|^2 = 0$.† When the quartic has a triple point, this breaks into a cubic Φ and the triple point counted three times. In the present case, the question asked above is most conveniently treated by the use of the polar conic of a point on the curve, as will now be shown.

The necessary and sufficient condition that a binary quartic with distinct roots have $g_3 = 0$, is that it contain as a factor the quadratic polar of each of

*The four points on a line section of a rational quartic are not in general projective with the binary quartic of their parameters; but when the curve has a triple point, the parameters are projective with lines on the triple point, which are again projective with the points of a line section.

† Clebsch, "Vorlesungen über Geometrie," p. 280. g_3 in Salmon's notation is the $C_{3,0}$ of the quartic.

its roots. To prove this, let the quartic be written $(\alpha t)(\beta t)(\gamma t)^2$, where α and β are real and γ symbolic. For the quadratic polar of (αt) we have

$$\begin{aligned}[(\alpha t)(\beta t)(\gamma t)^2,\ (\alpha t)^2] &= 2|\beta\alpha|\,|\gamma\alpha|\,(\alpha t)(\gamma t)+|\gamma\alpha|^2(\alpha t)(\beta t)\\ &=(\alpha t)\{2|\beta\alpha|\,|\gamma\alpha|\,(\gamma t)+|\gamma\alpha|^2(\beta t)\}.\end{aligned}$$

Since $|\beta\alpha|\neq 0$, (βt) becomes a factor when, and only when, $|\gamma\alpha|\,|\gamma\beta|=0$; *i.e.*, when $(\alpha t)(\beta t)$ is harmonic to $(\gamma t)^2$. Hence, the polar conic of a point t on the curve cuts it in points whose joins with t make harmonic line sections. This holds for the general rational quartic. If the curve has a triple point, three intersections of conic and curve are at the triple point (which proves that the contravariant of the preceding paragraph contains the triple point three times); two are at t and are to be disregarded, since the line section made by the tangent has a double root; the remaining three, when joined to t, give the lines of the cubic Φ on that point. We ask that a point t_1 with two of these points, say t_2 and t_3, form a mutually related set; *i.e.*, if t_1 gives t_2 and t_3, t_2 shall in the same way give t_1 and t_3, etc. Now the polar conic of a point y as to Θ is

$$\sum^3 a_0y_0(y_2-y_1)(x_2-x_1)^2+\sum^3 a_0x_0(x_2-x_1)(y_2-y_1)^2=0. \tag{26}$$

Let the functions (23) of t_1 and t be substituted for y and x respectively. After cancellation of the triple-point parameters and a factor $(t-t_1)^2$ corresponding to the point of contact of conic and curve, the result factors thus:

$$\begin{aligned}&[a_0t_1t(t+t_1-tt_1)+a_1(1-t_1)(1-t)(t_1t-1)+a_2(1-t_1-t)]\\ &\qquad\cdot[a_1a_2(t_1+t_2-2)-a_0a_2(t+t_1)+a_0a_1(t+t_1-2t_1t)]=0.\end{aligned} \tag{27}$$

Let these factors in order be denoted by ϕ_1 and ϕ_2. ϕ_1 is recognized as the quadratic polar of t_1 as to

$$C_{10,4}=[C_{7,3},\ C_{3,3}]^1=a_0^3a_1^3a_2^3\{a_0t^3(t-2)+a_1(1-t)^3(1+t)+a_2(2t-1)\}, \tag{28}$$

which is a self-apolar quartic. For the binary quartic $(\alpha t)^4\equiv(\beta t)^4\equiv(\gamma t)^4$, the condition that $(\alpha t_1)^2(\alpha t_2)^2=0$, $(\alpha t_1)^2(\alpha t_3)^2=0$ and $(\alpha t_2)^2(\alpha t_3)^2=0$ be simultaneously true is $|\alpha\beta|^4(\gamma t_1)^4=0$;* hence, *the necessary and sufficient condition that t_1 and the roots of its quadratic polar be mutually related for every t_1 is that $(\alpha t)^4$ be self-apolar.* (29)

Since $C_{10,4}$ is self-apolar, we have proved a characteristic property of Θ:

Through any point of the curve are two lines making harmonic sections with the curve and such that the line joining two of their further intersections also makes a harmonic section.

There are thus ∞^1 triangles of the kind sought. These will be called Φ triangles. On each side of such a triangle are two further intersections whose

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*Since t_2 and t_3 are roots of $(\alpha t_1)^2(\alpha t)^2=0$, we have $t_2t_3=\dfrac{a_1^2(\alpha t_1)^2}{a_0^2(\alpha t_1)^2}$, $t_2+t_3=\dfrac{-2a_0a_1(\alpha t_1)^2}{a_0^2(\alpha t_1)^2}$. Eliminate t_2 and t_3 from $(\alpha t_2)^2(\alpha t_3)^2\equiv[\beta_0^2t_2t_3+\beta_0\beta_1(t_2+t_3)+\beta_1^2]^2=0$. The result is $|\beta\alpha|^2|\beta\gamma|^2(\alpha t_1)^2(\gamma t_1)^2=0$, which must be $|\alpha\beta|^4(\gamma t_1)^2$ since $(\alpha t)^4$ has no other $C_{3,4}$.

parameters must satisfy $\phi_2=0$. ϕ_2 is the polarized form of $C_{2,2}\equiv a_1a_2(1-t)+a_0a_2t-a_0a_1t(1-t)$. For each t_1 this factor gives a unique line of Φ and could be used to obtain a parametric representation of Φ.

We consider next the involution on Θ cut out by conics on its triple point.* The base-points of this involution are given by a quintic covariant of f which we call $\bar{f}$ and whose covariants are indicated in like manner. The points $\bar{f}=0$ are cut out by a covariant conic on the triple point. Since a conic consisting of two lines on the triple point cuts out two arbitrary points, $C_{7,3}$ is the canonizant of $\bar{f}$. (Invariant factors are neglected when the function is to be equated to zero.) Line sections of Θ are quartics apolar to $\bar{f}$. Since a root of the Hessian of a quintic is a triple root of an apolar quartic, $\overline{H}=0$ gives the parameters of the flex-points, and $\overline{S}=0$ (the Steinerian of $\bar{f}$) the further intersections of the inflexional tangents with the curve. When a line section for which $g_3=0$ has a double root, it has a triple root, and the line is a flex-line. We therefore obtain the flex-parameters by putting $t=t_1$ in ϕ_1 and ϕ_2. It follows that $\overline{H}$ is $\rho C_{2,2}C_{10,4}$, where ρ is an invariant function or numerical. We now have these relations between the covariants of f and of $\bar{f}$:

$$C_{7,3}=\rho\overline{C}_{3,3}, \qquad C_{2,2}=\rho\overline{C}_{6,2}, \qquad C_{2,2}C_{10,4}=\rho\overline{H}.$$

Since $C_{2,2}$ is the Hessian of $C_{7,3}$ (§ 3), $\bar{f}$ is such that its Hessian contains its $C_{6,2}$ as a factor. Referring to the syzygy

$$6HC'-9i\tau^2-j(4jB+6\tau\alpha)=0$$

derived by Dr. A. E. Landry,† it is evident that when j (the canonizant) does not contain τ ($C_{6,2}$) as a factor, *i. e.*, when the discriminant of the canonizant does not vanish, $B=0$ is the necessary and sufficient condition that H contain τ. Hence, $\overline{B}=0$. (This B is the invariant of degree 8.)

It may be verified at once that the section made by the line $(1,1,1)$, which is the polar of the triple point as to the triangle ABC, is $C_{10,4}=0$. We have thus four points of inflexion on a line. The remaining flexes are given by $C_{2,2}=0$. We wish now to show that the flex-tangents at these points meet on the curve. Recurring to the expression ϕ_1, (27), it can be shown that t_1 with the roots t_2 and t_3 forms a cubic apolar to both $C_{3,3}$ and $C_{7,3}$. It is necessary first to prove a theorem on apolar cubics.

When two binary cubics are apolar, their Jacobian is a self-apolar quartic, and triads apolar to both cubics are made up of a point and its quadratic polar as to the Jacobian.

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* Clebsch, "Vorlesungen," p. 461.

† "A Geometrical Application of Binary Syzygies," *Transactions of the American Mathematical Society*, Vol. I (1909), p. 106.

Let the cubics be $(\alpha t)^3$ and $(\beta t)^3$, where $|\alpha\beta|^3=0$. Their Jacobian is $|\alpha\beta|(\alpha t)^2(\beta t)^2$. If t_1 is a root of a cubic apolar to $(\alpha t)^3$ and $(\beta t)^3$, the other two roots are given by the Jacobian of the quadratics $(\alpha t_1)(\alpha t)^2$ and $(\beta t_1)(\beta t)^2$; namely, $|\alpha\beta|(\alpha t_1)(\beta t_1)(\alpha t)(\beta t)$. We wish to prove that this quadratic is the same as the quadratic polar of t_1 as to $|\alpha\beta|(\alpha t)^2(\beta t)^2$. Polarizing $|\alpha\beta|(\alpha t)^2(\beta t)^2$ twice, we have

$$(1) \qquad |\alpha\beta|\{4(\alpha t)(\alpha t_1)(\beta t)(\beta t_1)+(\alpha t)^2(\beta t_1)^2+(\alpha t_1)^2(\beta t)^2\}=0.$$

We have identically:

$$(2) \qquad (\alpha t)(\beta t_1)\equiv|\alpha\beta|\,|t_1 t|+(\beta t)(\alpha t_1),$$

$$(3) \qquad (\alpha t_1)(\beta t)\equiv|\alpha\beta|\,|t t_1|+(\beta t_1)(\alpha t).$$

Substitute in the second and third terms of (1) after multiplying (2) and (3) by $(\alpha t)(\beta t_1)$ and $(\alpha t_1)(\beta t)$ respectively. The result, after simple reductions, is

$$(4) \qquad 6|\alpha\beta|(\alpha t)(\alpha t_1)(\beta t)(\beta t_1)+|\alpha\beta|^3|t_1 t|^2=0.$$

Since $|\alpha\beta|^3=0$, this reduces to its first term, and the second part of the theorem is proved. To see that the Jacobian is a self-apolar quartic, it is sufficient to note that the roots of a cubic apolar to two cubics are in an involution and to apply (29).

Now $C_{3,3}$ and $C_{7,3}$ are apolar, and $C_{10,4}$ is their Jacobian. We have thus proved that *the* ∞^1 *cubics which give vertices of* Φ *triangles are the self-apolar pencil* $C_{3,3}+\lambda C_{7,3}$. (30)

The product $C_{2,2}C_{5,1}$ is a member of this pencil; for, $[C_{3,3}, C_{2,2}]^2\equiv C_{5,1}$ and $[C_{7,3}, C_{2,2}]^2\equiv 0$. Hence, the triangle $C_{2,2}C_{5,1}$ is a Φ triangle. Since the polar conic of a point of inflexion is the flex-tangent and a line on the triple point, two Φ lines through a flex-point coincide with the flex-tangent, which is therefore a side of the Φ triangle. Hence, when two flex-points are vertices, the flex-lines must meet at the third vertex, which is a point of the curve. We have now proved a second important property of Θ:

The six inflexions break up into sets of four and two; the four are on a line, and the tangents at the remaining two meet on the curve. The parameters of the two sets are given by $C_{10,4}=0$ *and* $C_{2,2}=0$ *respectively.*

The intersection of the two flex-lines, the point $C_{5,1}$ on Θ, is on the line joining $C_{2,2}$ and $C_{6,2}$ as points determined by quadratics on N; for, $[C_{2,2}C_{3,3}]^2\equiv C_{5,1}$, which says that the line η of $C_{5,1}$ passes through point $C_{2,2}$. Lines through $C_{5,1}$ on Θ cut the curve in three further points which form a pencil $C+\lambda C'$, where C is a cubic and C' its cubic covariant.

We consider now tangents to Θ at vertices of a Φ triangle. Tangents at the reference points are easily seen to meet at point $C_{2,2}$; tangents at the vertices $C_{2,2}C_{5,1}$ meet at $C_{5,1}$; and tangents at the triple point $C_{7,3}$ (a Φ triangle with

coincident vertices) meet there. It will be shown that the tangents at vertices of each Φ triangle meet at a point, and that the locus of such points is the straight line joining $C_{2,2}$ and $C_{6,2}$. Beginning from the converse standpoint, we find the sextic of points of contact of tangents from a point x of the plane. The tangent at y on the curve is

$$3\overset{3}{\Sigma} a_0 y_0 (y_2-y_1)^2 (x_2-x_1) + \overset{3}{\Sigma} a_0 x_0 (y_2-y_1)^3 = 0.$$

Substituting for y the functions (23), the result reduces to

$$3C_{3,3}C_{2,2}a_0a_1a_2[x_0t+x_1(1-t)-x_2]+C_{7,3}[a_0t^3x_0+a_1(1-t)^3x_1-a_2x_2]=0, \qquad (31)$$

which gives the points of contact of tangents from x. When x is on the join of $C_{6,2}$ (or $1, 1, 1$) and $C_{2,2}$ $(a_0^{-1}, a_1^{-1}, a_2^{-1})$, we have $x_i=a_i^{-1}+\mu$, $i=0, 1, 2$. Substituting these values of x in (31), the sextic becomes

$$3a_0a_1a_2C_{3,3}C_{2,2}\left[\frac{t}{a_0}+\frac{1-t}{a_1}-\frac{1}{a_2}\right]+C_{7,3}[t^3+(1-t)^3-1]$$
$$+\mu C_{7,3}[a_0t^3+a_1(1-t)^3-a_2]=0.$$

In terms of covariants this is

$$3C_{3,3}[C_{2,2}C_{5,1}-C_{7,3}]+\mu(a_0a_1a_2)^{-1}C_{7,3}^2=0.$$

Since $C_{4,0}C_{3,3}\equiv C_{2,2}C_{5,1}-C_{7,3}$, we obtain

$$3[C_{2,2}C_{5,1}+C_{7,3}][C_{2,2}C_{5,1}-C_{7,3}]+\mu(a_0a_1a_2)^{-1}C_{4,0}C_{7,3}^2=0,$$

or

$$3C_{2,2}^2C_{5,1}^2-[3-(a_0a_1a_2)^{-1}\mu C_{4,0}]C_{7,3}^2=0.$$

The factors of this are of the form $C_{2,2}C_{5,1}+\lambda C_{7,3}$, which is identical with the pencil (30) giving the vertices of Φ triangles. Hence:

The six points of contact of tangents drawn from a point of the line $C_{6,2}C_{2,2}$ break into two sets of three, and each set forms a Φ triangle.

It is interesting to note that Φ triangles may be obtained by drawing the lines of K (§ 2) from a point on the line $C_{6,2}C_{2,2}$ and taking their intersections with the corresponding lines η (§ 5, (21)). For, if the line K is made to pass through $x_i=a_i^{-1}+\mu$, the resulting cubic is $3a_0a_1a_2C_{3,3}-\mu C_{7,3}$, which is again the pencil (30). Triads of this pencil on N are points of contact of triangles circumscribed about N and inscribed in the conic F on the five points A, B, C, $C_{2,2}$ and $C_{6,2}$. For, the conic on the vertices of triangles whose sides touch N at $C_{3,3}=0$ and $C_{7,3}=0$ is on the Hessian point of each cubic. The conic F cuts N in $C_{10,4}=[C_{3,3}, C_{7,3}]^1$.

The triangle $C_{2,2}C_{5,1}$ on Θ (formed by the two flex-lines which meet on the curve and the line joining their flex-points) is a natural triangle of reference for the curve. Let the triple point be the unity point as before, and take the line joining the flexes as $x_1=0$. The curve may be generated by the inter-

section of a line on $(1, 1, 1)$ with a correlated line on $(0, 1, 0)$, the intersection of the flex-lines. The coordinates of the variable lines may be taken as

$$\left.\begin{array}{llll}\eta: & t, & 1-t, & -1,\\ \xi: & at^3, & 0 & -b.\end{array}\right\} \tag{32}$$

The curve is then written

$$\left.\begin{array}{c} x_0=b(1-t), \qquad x_1=at^3-bt, \qquad x_2=at^3(1-t),\\ ax_0(x_2-x_1)^3+bx_2(x_1-x_0)^3=0.\end{array}\right\} \tag{33}$$

The reference scheme uses eight coordinates, and the ratio $a:b$ gives the nine necessary to determine the curve. Since three line sections are known, the fundamental quintic to which all line sections are apolar is readily found:

$$\bar{f}=a(t^5-5t^4+10t^3)-b(10t^2-5t+1),$$

from which $\bar{B}=I_8=0$ is verified. The conic on the points $\bar{f}=0$ is

$$x_0^2+6x_1^2+x_2^2+3x_1(x_0+x_2)-14x_0x_2=0.$$

The parameters of the four flexes on a line were given by the Jacobian of cubics of any two Φ triangles. We now have $\bar{C}_{3,3}=C_{7,3}=at^3-b$ and $C_{2,2}C_{5,1}=t(1-t)$. The Jacobian is $b-2t+2at^3-at^4$, which is again cut out by the line $(1, 1, 1)$. This verifies (what could be seen otherwise):

The line on four flexes is the polar of the triple point as to all Φ triangles. Moreover, *the conic on $\bar{f}=0$ is tangent to this line.*

Φ is now $a_0\xi_0(\xi_1+\xi_2)^2-b\xi_2(\xi_0+\xi_1)^2=0$, and the locus of self-apolar line sections is $4\xi_0\xi_1-(\xi_1+\xi_2)(\xi_0+\xi_1)=0$. Parametric equations of Φ are obtained by making use of the fact that when t_1 is a vertex of a Φ triangle the quadratic of intersections other than vertices on the opposite side is the quadratic polar of t_1 as to $C_{7,3}$. This is proved by consideration of the values $t_1=0, 1, \infty$. The quadratic is $at_1t^2-b=0$. Using (33), the line joining the two points on Θ given by this quadratic is found to have coordinates

$$\xi_0=-b(1-t), \qquad \xi_1=t(at-b), \qquad \xi_2=-at^2(1-t),$$

the subscript having been dropped from t_1. These equations express a side of a Φ triangle in terms of the parameter t of the opposite vertex.

We can now prove a property of the six points of the curve other than vertices on the sides of a Φ triangle.* It has been seen that the two points on one side of the triangle are given by the quadratic $at_1t^2-b=0$. Calling the roots t and $-t$, we write in line coordinates the points t and $-t$ of Θ given by (33). The product of these points is rational in t^2. Eliminating t^2 by means of $at^2-b=0$, we have, for the points on the side opposite t_1,

*This property is made the basis of the discussion of the curve by R. A. Roberts (*Proceedings of the London Mathematical Society*, Vol. XVI).

$$a\xi_0^2 t_1^3(at_1-b)-ab\xi_1^2 t_1(1-t_1)^2-b\xi_2^2(at_1-b)$$
$$+2\xi_0\xi_1 abt_1^2(1-t)-2\xi_1\xi_2 abt_1(1-t_1)=0. \qquad (34)$$

We wish to show that three such degenerate line conics are members of a pencil when t_1, t_2 and t_3 are vertices of a Φ triangle. t_1, t_2 and t_3 must satisfy an equation of the form $C_{2,2}C_{5,1}+\lambda C_{7,3}=0$. In the present notation this is $at^3-\lambda t^2+\lambda t-b=0$. The conditions are therefore (the s's being the symmetric functions of the three t's) $s_1=s_2$ and $s_3=b/a$. The second condition is used to eliminate b from (34). After division by a^2, the result is

$$\xi_0^2 t_1^2(t_1-s_3)-\xi_1^2 t_1(1-t_1)^2-\xi_2^2(t_1-s_3)+2\xi_0\xi_1 t_1^2(1-t_1)-2\xi_1\xi_2 t_1(1-t_1)=0.$$

The matrix of three such conics, when $s_1=s_2$, vanishes identically, and the conics therefore belong to a pencil. Hence:

The six further points of the curve on the sides of a Φ triangle are six points on four lines.

§ 7. *The Binary Quintic Represented on a Rational Cubic in Space.*

Let the ρ^3 in space be written in points:

$$x_0=t^3, \qquad x_1=3t^2, \qquad x_2=3t, \qquad x_2=1, \qquad (35)$$

and in planes:

$$\xi_0=1, \qquad \xi_1=-t, \qquad \xi_2=t^2, \qquad \xi_3=-t^3. \qquad (36)$$

Any point of space in terms of parameters of planes from it to the curve is

$$x_0=s_3, \qquad x_1=s_2, \qquad x_2=s_1, \qquad x_3=1; \qquad (37)$$

and any plane in terms of its intersections with the curve is

$$\xi_0=1, \qquad \xi_1=-\tfrac{1}{3}s_1, \qquad \xi_2=\tfrac{1}{3}s_2, \qquad \xi_3=-s_3, \qquad (38)$$

where s_1, s_2 and s_3 are the elementary symmetric functions of three t's.

Let the quintic be written in canonical form as before (§ 1, (1)), and let the canonizant points A, B, C $(t=0, 1, \infty)$ determine a plane π and a point p. From (37) and (38) it is seen that the condition that two cubics be apolar is the same as the condition that the point of each lie on the plane of the other. Since a cubic is self-apolar, p is on π. π is the plane $x_1=x_2$, and p has coordinates $(0, 1, 1, 0)$.

From any point of π are three planes of ρ^3 whose intersections with π are lines of a rational plane cubic. At A, B and C the three planes coincide; hence, the curve has three cusps and the intersection of the cusp-tangents is p. It is a rational quartic in points. Since the intersection of two consecutive planes of ρ^3 is the tangent line, the same curve is the locus of the intersection of tangent lines of ρ^3 with π. It is obviously the section of the developable quartic surface which ρ^3 determines, and is independent of the ratios $a_0:a_1:a_2$. To find its double line, we look for a line of π which contains two planes of ρ^3.

The bisecant line from p meets ρ^3 in the Hessian of the canonizant cubic. Since these two t's and any t are apolar to $C_{3,3}$, the planes of these points and any plane of ρ^3 meet on π; that is, the two planes meet on π and give the double line. The points of contact are cut out by the two tangent lines to ρ^3. The parameters are given by $C_{6,2}=0$.

A transformation of the plane is determined by quartics on ρ^3 apolar to f, as follows: let $t_1t_2t_3t_4$ be apolar to f, and let lines t_1t_2 and t_3t_4 meet π in P and Q, respectively; then P and Q are corresponding points of the transformation. That this transformation is one-to-one and involutory is clear from the fact that from a point of the plane but one bisecant line can be drawn, and that the two values of t thus found uniquely determine the second pair in the apolar quartic. It is the same transformation as is gotten by taking the cubic polar of t_1t_2 as to f and finding the intersection of planes of ρ^3 at these points. Since cubic polars are apolar to $C_{3,3}$, the point so found will lie in π. To express analytically this transformation, we take as reference triangle the triangle ABC, and as unity point the transform of p. If the bisecant line from P of π meets ρ^3 at t_1 and t_2, the transformed point Q is given by the cubic

$$a_0t^3t_1t_2+a_1(1-t)^3(1-t_1)(1-t_2)-a_2=0.$$

The coordinates of Q are therefore

$$\left.\begin{aligned} x_0&=s_3=-a_1(1-t_1)(1-t_2)+a_2,\\ x_1&=x_2=s_2=-3a_1(1-t_1)(1-t_2),\\ x_3&=a_0t_1t_2-a_1(1-t_1)(1-t_2). \end{aligned}\right\} \tag{39}$$

Now the bisecant from p gives points $C_{6,2}=0$ on ρ^3; *i. e.*, t_1 and t_2 satisfy $t^2-t+1=0$. Using these values in (39), we find $x_0=a_1-a_2$, $x_1=x_2=3a_1$, $x_3=a_1-a_2$. This is the transform of p, and is to be taken as the unity point $y_0=y_1=y_2$ in the plane. Since the space coordinates of A, B, and C are known from (35), the transformation from space to plane is readily found:

$$\left.\begin{aligned} x_0&=a_1y_1-a_2y_2, & a_0y_0&=(x_1-3x_3),\\ x_1&=x_2=3a_1y_1, & a_1y_1&=x_1, \qquad x_1=x_2,\\ x_3&=a_1y_1-a_0y_0, & a_2y_2&=(x_1-3x_0). \end{aligned}\right\} \tag{40}$$

Applying this substitution to the equations (39), we find for Q:

$$y_0=t_1t_2, \qquad y_1=(1-t_1)(1-t_2), \qquad y_2=1. \tag{41}$$

It is thus proved that *if the bisecant line from P meets ρ^3 in t_1t_2, the transformed point Q is at the intersection of tangents t_1t_2 to the conic $y_0=t^2$, $y_1=(1-t)^2$, $y_2=1$.* The position of the conic is determined by the reference scheme as described above. The coordinates of P in terms of t_1t_2 are found from the equations of two planes on the chord t_1t_2 of ρ^3. Changing to y coordinates by (40), we find for the intersection of the chord with π:

$$y_0' = [a_0 t_1 t_2]^{-1}, \qquad y_1' = [a_1(1-t_1)(1-t_2)]^{-1}, \qquad y_2' = [a_2]^{-1}. \tag{42}$$

Comparison with (41) shows that P and Q satisfy (17). Hence, *the transformation of the present paragraph is identical with the quadratic transformation T of* § 3.

When $t_1 = t_2$, P traces the quartic K and Q the conic N (see § 2, following (13)). Point p named by its tangent to N is $C_{2,2}=0$, and its transform $(1,1,1)$ is $C_{6,2}=0$ as before. If from the fixed points of T in π the bisecants are drawn to ρ^3, the four quadratics $(\alpha t_1)^3(\alpha t)^2=0$, $(\alpha t)^3(\alpha t_1)^2=0$, or $C_{4,8}=0$ (14), are obtained. The quadric cone formed by projecting ρ^3 from a point t on it cuts π in a conic on ABC which is the transform of line t on N.

The rational plane quintic O (§ 4) was defined as the locus of the intersection of a tangent to N with the corresponding line of K (§ 2). For this we may now substitute the following: Given ρ^3 and a plane cutting it, inscribe any

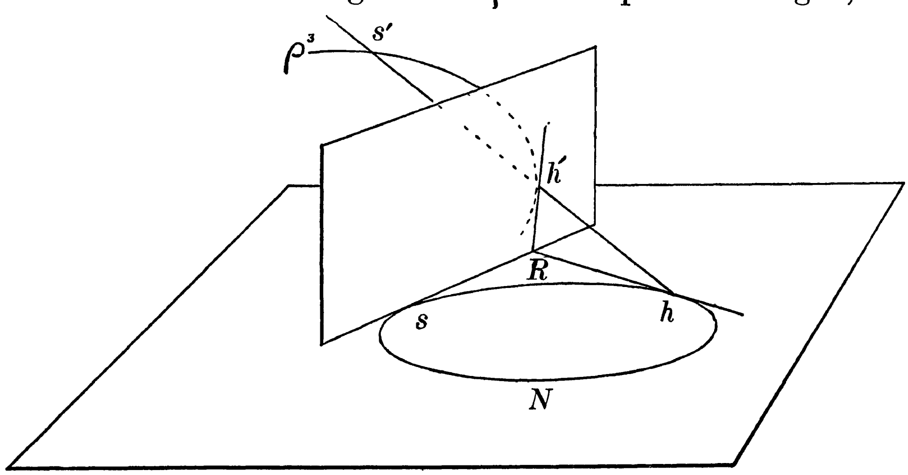

FIG. 2.

conic in the triangle ABC of the three points in the plane; correlate points of conic and cubic in such a way that the point of contact of the conic with the side BC corresponds to A on ρ^3, etc.; the plane t of ρ^3 and the tangent to the conic at the corresponding point t meet at a point whose locus is O. The number of coordinates used in the plane is 8 for the triangle and p, and 2 for the conic. The binary quintic is not mentioned, but it is at once recovered through the fact that p gives $C_{2,2}$ on N, and $(1,1,1)$ gives (by its bisecant) $C_{2,2}$ on ρ^3.

Certain relations between the points given by the Hessian and Steinerian covariants on ρ^3 and N are of interest. (α) Let h and s be a pair of corresponding roots of the Hessian and Steinerian of f on N, and h' and s' the points having the same parameters on ρ^3. (See Fig. 2.) By the definition of K (§ 2) the line of K having h as parameter is tangent to N at S; and it has been shown in the present section that a line t of K is the intersection of π

with the plane t of ρ^3. Hence, *the six planes of* ρ^3 *at the points* $H=0$ *cut the plane of the canonizant in six lines of a conic, and the points of contact are the points* $S=0$ *of a correlated quintic on the conic.* (β) The tangent to ρ^3 at h' meets π at R, the point with parameter h on K. This point has been shown to be the intersection of tangents to N at s and h (§ 4, first paragraph). If we join s' to a variable point t' on ρ^3, the line meets π in a conic whose transform by T is the line Rs, as was shown above; this is in accord with the fact that the cubic polar of $s't'$ always has a root at h'. When $t'=h'$, the cubic polar has a second root at h', and the point determined on π by this cubic is on the intersection of two consecutive planes at h'; that is, the point is R. Now the chord $s't'$ meets π in the transform by T of the point given by the cubic polar of $s't'$. Since h^3s is a quartic apolar to f, the transform of R is h on N. Hence, chord $s'h'$ contains h. *If the points* h *on* N *were given, we should obtain the pairs* $s'h'$ *by drawing bisecants to* ρ^3 *from them; the planes of* ρ^3 *at points* h' *will then determine points* s *on* N. (γ) Since Rs is tangent to N and K, the conic into which Rs transforms by T is tangent to K and N: to N at h, and to K at the transform of s. Therefore, *six conics on* ABC *are tangent to* K *and* N; *the points of contact on* N *are at* $H=0$, *and the points of contact on* K *are cut out by tangents to* ρ^3 *at* $S=0$.

Two parameters t_1, t_2 on ρ^3 give a cubic polar whose plane (named by t_1, t_2) passes through p. The locus of the intersection of chord t_1t_2 with plane t_1t_2 is a surface Ω whose points are in one-to-one correspondence with the points of π, the point t_1t_2 of the surface being made to correspond to the point y of (41). (It will be recalled that this y is the transform by T of the intersection of chord t_1t_2 with π.) Let

$$t_1+t_2=\sigma_1, \qquad t_1t_2=\sigma_2. \tag{43}$$

Equations (41) are equivalent to

$$y_0=\sigma_2, \qquad y_1=1-\sigma_1+\sigma_2, \qquad y_2=1. \tag{44}$$

Two planes on the chord are

$$\left.\begin{aligned} 3x_0-\sigma_1x_1+\sigma_2x_2 \qquad &=0,\\ -x_1+\sigma_1x_2-\sigma_2x_3&=0.\end{aligned}\right\} \tag{45}$$

Using (10) with change of subscripts and (38), the plane t_1t_2 is found to be

$$a_0\sigma_2x_0+a_1(1-\sigma_1+\sigma_2)(-x_0+x_1-x_2+x_3)-a_2x_3=0. \tag{46}$$

Elimination of σ_1 and σ_2 from (45) and (46) gives the equation of the surface. To express analytically the above-mentioned correspondence between points of Ω and π, we eliminate σ_1 and σ_2 between (44), (45) and (46). Or, eliminating x from the equations (45) and (46) and $(x\xi)=0$, then using (44), we find as the equation of the point of Ω corresponding to y:

$$\begin{vmatrix} 3y_2 & -(y_0-y_1+y_2) & y_0 & 0 \\ 0 & -y_2 & y_0-y_1+y_2 & -3y_0 \\ a_0y_0-a_1y_1 & a_1y_1 & -a_1y_1 & a_1y_1-y_2y_2 \\ \xi_0 & \xi_1 & \xi_2 & \xi_3 \end{vmatrix} = 0. \qquad (47)$$

The coefficients of the ξ's in this equation are the coordinates of x; that is, we have $x_i = f_i(y_0, y_1, y_2)$, where f_i are homogeneous cubic functions of y. *The transformation from plane to surface is then such that a plane section $(x\xi)=0$ is the map of the plane cubic $(f\xi)=0$.*

The four cubics $f_i(y_0, y_1, y_2)=0$ have six points in common; they are in fact cubic adjoints of O (§ 4). To see this, we recall that the parameters of a double point of O are subject to the conditions $(\alpha t_1)^4(\alpha t_2)=0$ and $(\alpha t_2)^4(\alpha t_1)=0$. If t_1 and t_2 are roots of $(\beta t)^2=0$, this is equivalent to the requirement that $(\beta t)^2$ be contained as a factor in its cubic polar $|\alpha\beta|^2(\alpha t)^3$. In Salmon's notation this becomes

$$a_0t_1t_2t^3 + a_1(1-t_1)(1-t_2)(1-t)^3 - a_2 \equiv \rho(b_0+2b_1t+b_2t^2)(c_0+c_1t).$$

Eliminating c_0, c_1 and ρ from the four equations to which this identity gives rise, and substituting for the b's their equivalents in terms of the point y determined by the quadratic, we obtain four equations in y which a double point must satisfy. These are precisely the four cubics found above.

The surface Ω contains ρ^3, and this curve is the map of O; for, if t_1 and t satisfy $(\alpha t_1)^4(\alpha t)=0$, the cubic polar of the quadratic tt_1 contains t_1 as a factor, and the intersection of chord tt_1 with plane tt_1 is the point t_1 on ρ^3. The corresponding point tt_1 of π is on O by definition of the curve.

Surface Ω meets π when chord t_1t_2 meets plane t_1t_2 on π. Let chord t_1t_2 meet π at P as before. The plane t_1t_2 (*i. e.*, the plane cutting ρ^3 in the cubic polar of t_1t_2) contains Q, the transform of P by T, and p. Therefore, the surface meets π when P and Q are in line with p. The locus of such points is the plane cubic Ω. (See § 5. Point $C_{2,2}$ of § 5 is p.) Making $x_1=x_2$ in the equation of the surface and using the transformation (40) gives equation (20) of § 5. That A, B and C, the fixed points of T, and p are on this curve is seen at once from the present standpoint. The double points of O were given by quadratics which are factors in their cubic polars. If t_1t_2 is such a quadratic on ρ^3, the plane of the cubic polar contains the line t_1t_2; P and Q are then on the section of the plane with π, which says that the double points Q are on curve Ω. Since the line t_1t_2 just referred to lies in its plane, it is a line of the surface Ω. Six lines of the surface are thus accounted for. The six conics and fifteen lines determined by the double points of O map into the remaining twenty-one lines.

A theorem proved by Stahl* states that adjoint cubics of a rational quintic which are on two fixed points t_1 and t_2 of the curve meet further at the intersection of tangents to the perspective conic with the same parameters. In the above mapping process, the point of π whose parameters with respect to the conic N are t_1 and t_2, was made to correspond to the third intersection with Ω of the chord t_1t_2 on ρ^3. *Stahl's theorem follows at once from the fact that cubics on two fixed points of O map into plane sections on a fixed chord of ρ^3 which have necessarily the same third intersection with Ω.*

§ 8. *Further Consideration of Rational Quintic O of § 4.*

The quintic O was defined as the locus of the intersection of tangents to N at points t and t' when $(\alpha t)^4(\alpha t')=0$. It was seen that the parameters of its double points satisfy $(\alpha t)^4(\alpha t')=0$ and $(\alpha t')^4(\alpha t)=0$, from which, by elimination of t', we obtain $C_{1,5}\cdot C_{4,12}=0$. The first factor is accounted for by the fact that O is five times tangent to N at the points $f=0$ (§ 4). The parameters of the double points then satisfy a $C_{4,12}=0$.

To express this $C_{4,12}$ in terms of the irreducible system, we let 0 and ∞ be two of its roots. Then f takes the form

$$f\equiv\alpha_0t^5+10\alpha_2t^3+10\alpha_3t^2+\alpha_5\,. \tag{48}$$

$C_{4,12}$ must be of the form $C^2_{2,6}+\lambda C_{2,2}C^2_{1,5}$. We determine λ so that this shall have roots 0, ∞ when $\alpha_1=\alpha_4=0$. In symbols,

$$C_{2,6}=|\alpha\beta|^2(\alpha t)^2(\beta t)^2, \qquad C_{2,2}=|\alpha\beta|^4(\alpha t)(\beta t), \qquad C_{1,5}=(\alpha t)^5.$$

Putting $t=0$ gives for the determination of λ:

$$[\,|\alpha\beta|^2\alpha_0^3\beta_0^3]^2+\lambda|\alpha\beta|^4\alpha_0\beta_0(\alpha_0^5)^2=0,$$

whence, when $\alpha_0^4\alpha_1=\alpha_1^4\alpha_0=0$, $\lambda=-\frac{2}{3}$. Hence,

$$C_{4,12}=C^2_{2,6}-\tfrac{2}{3}C_{2,2}C^2_{1,5}\,. \tag{49}$$

The above form (48) of f isolates one of the six double points by giving it the parameters $(0, \infty)$. Certain properties of the curve are easily discussed from this standpoint. We take N in the form

$$x_0=t^2, \qquad x_1=2t, \qquad x_2=1. \tag{50}$$

The triangle of reference is now formed by two tangents from a double point to N and the line joining their points of contact. The equation of the conic on the remaining five points is found as follows: The adjoint cubics, found by the method of § 7 (paragraph following (47)), now take the form

$$\begin{vmatrix} \alpha_0x_0+\alpha_2x_2 & 3(\alpha_2x_1+\alpha_3x_2) & 3(\alpha_2x_0+\alpha_3x_1) & \alpha_3x_0+\alpha_5x_2 \\ x_2 & -x_1 & x_0 & 0 \\ 0 & x_2 & -x_1 & x_0 \\ \eta_0 & \eta_1 & \eta_2 & \eta_3 \end{vmatrix}=0. \tag{51}$$

* "Zur Erzeugung der ebenen rationalen Curven," *Mathematische Annalen*, Vol. XXXVIII (1891).

By proper choice of the η's, this will break into a conic and a line on the double point $(0,1,0)$; for convenience, we make it contain x_0 as a factor. The resulting equations for the η's are

$$4\alpha_2\eta_3-\alpha_5\eta_0=0, \qquad 6\alpha_3\eta_3-\alpha_5\eta_1=0, \qquad \eta_2=0,$$

from which is obtained the conic

$$6\alpha_0\alpha_3x_0^2+(\alpha_0\alpha_5-16\alpha_2\alpha_3)x_1^2+6\alpha_2\alpha_5x_2^2-24\alpha_3^2x_1x_2 \\ -24\alpha_2^2x_0x_1-(20\alpha_2\alpha_3+\alpha_0\alpha_5)x_0x_2=0. \tag{52}$$

The product of the tangents from $(0,1,0)$ to conic (52) is found to be

$$x_0^2[6\alpha_0\alpha_3(\alpha_0\alpha_5-16\alpha_2\alpha_3)-144\alpha_2^4]+x_2^2[6\alpha_2\alpha_5(\alpha_0\alpha_5-16\alpha_2\alpha_3)-144\alpha_3^4] \\ +x_0x_2[32\alpha_2^2\alpha_3^2-\alpha_0^2\alpha_5^2-4\alpha_0\alpha_2\alpha_3\alpha_5]=0. \tag{53}$$

To find the cubic adjoints with double point at $(0,1,0)$, we make the polar conic of this point, as to cubic (51), degenerate into two lines of the form $ax_0^2+bx_0x_2+cx_2^2=0$. The resulting two conditions on η_i give $\eta_0=\eta_3=0$. The pencil of rational cubics is

$$\eta_1[x_0^2(\alpha_0x_0-2\alpha_2x_2)-x_1x_2(4\alpha_3x_0+\alpha_5x_2)] \\ +\eta_2[x_2^2(\alpha_5x_2-2\alpha_3x_0(-x_0x_1(\alpha_0x_0+4\alpha_2x_2)]=0. \tag{54}$$

Two members of the pencil have a cusp at $(0,1,0)$. To find the cusp tangents, we require that the polar conic

$$\alpha_0\eta_2x_0^2-\alpha_5\eta_1x_2^2-(4\alpha_3\eta_1-4\alpha_2\eta_2)x_0x_2=0$$

be a perfect square. Elimination of η_1 and η_2 between this quadratic and its discriminant gives for the cusp-lines:

$$2\alpha_0\alpha_3x_0^2+\alpha_0\alpha_5x_0x_2+2\alpha_2\alpha_5x_2^2=0. \tag{55}$$

There is a simple relation between these lines, the tangents to O at the same point, and the tangents to N. The equations for O are now:

$$\left.\begin{aligned} x_0&=4\alpha_2t^4+6\alpha_3t^3+\alpha_5t, \\ x_1&=-\alpha_0t^5-2\alpha_2t^3+2\alpha_3t^2+\alpha_5, \\ x_2&=-\alpha_0t^4-6\alpha_2t^2-4\alpha_3t. \end{aligned}\right\} \tag{56*}$$

The line equations are found from these by differentiation, and the values $t=0$, ∞ give for the tangents to O at its double point:

$$(4\alpha_3x_0+\alpha_5x_2)(\alpha_0x_0+4\alpha_2x_2)=0. \tag{57}$$

The same pair of lines are obtained by differentiating (55) partially with respect to x_0 and x_2. Hence, *the tangents to O are the harmonic conjugates of the tangents to N with respect to the cusp-lines of cubic adjoints, all six lines being drawn from the same double point of O.*

From the standpoint of the space cubic ρ^3 (§ 7), the above result is interpreted as follows: The six double points $(1, 2, \ldots, 6)$ of O map into six lines

*This is found as the intersection of line t of N with the corresponding line t of K. See §4. The coordinates of the lines are respectively $(1, -t, t^2)$ and $(a_0^2(at)^3, a_0a_1(at)^3, a_1^2(at)^3)$.

of the surface Ω; and the conics $C_1, \ldots, C_6$ on five of the six points map into six lines, the two sets forming a double six. Let point 1 give line a_1, meeting ρ^3 at t_1t_2, and let the corresponding conic C_1 give line b_1 (which does not meet ρ^3). The cusp-directions (55), being the double lines of the involution of tangents to cubic adjoints, transform into the points of a_1 on the parabolic curve of Ω. Any line on point 1, in particular the tangent t_1 to N, transforms into a plane section on b_1. Since the transform of N is the section of Ω with the developable surface determined by ρ^3, the point of contact t_1 on N becomes the third intersection of tangent t_1 to ρ^3 with Ω. Hence, interpreting f on ρ^3 without regard to N:

If t_1, t_2 are a pair of associated roots of $C_{4,12}$, (49), *and give chord a_1 of ρ^3, and if b_1 is the corresponding line of the double six on Ω, the line t_1 of ρ^3 meets Ω in a further point P such that the plane b_1P cuts a_1 in the harmonic conjugate of t_1 as to the parabolic points of Ω on a_1.*

The five lines from (0, 1, 0) to the remaining double points are found by elimination of x_1 from the cubics in the brackets of (54). The result is

$$\alpha_0^2x_0^5+2\alpha_0\alpha_2x_0^4x_2-8\alpha_2^2x_0^3x_2^2+8\alpha_3^2x_0^2x_2^3-2\alpha_3\alpha_5x_0x_2^4-\alpha_5^2x_2^5=0. \tag{58}$$

The lines to the five points $f=0$ on N are found by eliminating t between $f=0$ and $x_0=t^2$, $x_2=1$:

$$\alpha_0x_0^5+20\alpha_0\alpha_2x_0^4\alpha_2+100\alpha_2^2x_0^3x_2^2-100\alpha_3^2x_0^2x_2^3-20\alpha_3\alpha_5x_0x_2^4-\alpha_5^2x_2^5=0. \tag{59}$$

Quintics (58) and (59) are apolar. That is, *the six points have the property that lines joining one to the other five are apolar to lines joining the same point to a set of five fixed points in the plane.*

We now have $C_{2,2}=3\alpha_2^2t^2+(\alpha_0\alpha_5+2\alpha_2\alpha_3t)+3\alpha_3^2$. The point $C_{2,2}$ therefore has coordinates

$$x_0=3\alpha_3^2, \qquad x_1=-(\alpha_0\alpha_5+2\alpha_2\alpha_3), \qquad x_2=3\alpha_2^2.$$

The cubic polar of the quadratic $(0, \infty)$ as to f is $3\alpha_2t^2+3\alpha_3t$. Two roots of this are $(0, \infty)$, and the third is the γ of § 4. The point γ on N is therefore

$$x_0=\alpha_3^2, \qquad x_1=-2\alpha_2\alpha_3, \qquad x_2=\alpha_2^2.$$

Comparison with the coordinates of $C_{2,2}$ shows that these points are in line with the double point (0, 1, 0), which agrees with the theorem of § 4. We have also the coordinates of the following lines:

Polar of $C_{2,2}$ as to N: $\quad 6\alpha_2^2, \quad (\alpha_0\alpha_5+2\alpha_2\alpha_3), \quad 6\alpha_3^2.$

Polar of (0, 1, 0) as to conic (52): $\quad 12\alpha_2^2, \quad (-\alpha_0\alpha_5+16\alpha_2\alpha_3), \quad 12\alpha_3^2.$

These lines meet on $x_1=0$, the polar of (0, 1, 0) as to N. Now the line $C_{2,2}$ is determined by a covariant of f and is therefore independent of the choice of the double point to be used as (0, 1, 0). Hence:

The intersections of the polars of each double point as to N, and as to the conics on the other five, are six points on a line.

The line on which the above six points lie (*i. e.*, the polar of $C_{2,2}$ as to N) is a covariant line of O which appears also in the case of the general rational plane quintic. It makes the unique line section apolar to all line sections.* Since O is defined $(\alpha t)^4(\alpha t')=0$, a point of the curve falls on the line $C_{2,2}=0$ when quadratic tt' is apolar to $C_{2,2}$. Hence, the parameters of points on this line satisfy $[f, C_{2,2}]=C_{3,5}=0$. In the present notation we have

$$C_{3,5}\equiv\alpha_0(\alpha_0\alpha_5+2\alpha_2\alpha_3)t^5+6(\alpha_0\alpha_3^2-4\alpha_2^3)t^4+2\alpha_2(\alpha_0\alpha_5-16\alpha_2\alpha_3)t^3$$
$$-2\alpha_3(\alpha_0\alpha_5-16\alpha_2\alpha_3)t^2-6(\alpha_2^2\alpha_5-4\alpha_3^3)t-\alpha_5(\alpha_0\alpha_5+2\alpha_2\alpha_3).$$

The apolarity of this with the line sections $x_i=0$, (56), is easily verified.

Parametric equations for the conic C_1 on double points 2, 3, 4, 5, 6 may be found by use of a corollary to Stahl's theorem on the rational quintic (see end of § 7). The corollary states that C_1 and the perspective conic N are in triangular relation. Consider triangles with vertices on C_1 and sides touching N. We name a point of C_1 by the t of the point of contact on the opposite side of the triangle. C_1 is given by equation (52). Line $x_0=0$ cuts C_1 in points whose equation is

$$(\alpha_0\alpha_5-16\alpha_2\alpha_3)\xi_2^2+6\alpha_2\alpha_5\xi_1^2+24\xi_1\xi_2=0.$$

To find points of contact of tangents from these points to N, write $\xi_0=1$, $\xi_1=-t$, $\xi_2=t^2$. The resulting quadratic gives the parameters of the points of C_1 on $x_0=0$. The quadratic $x_2=0$ is found similarly. The tangent to N at $t=1$ is $x_0-x_1+x_2=0$. The tangent at a variable point t, $x_0-x_1t+x_2t^2=0$, meets this at $x_0=t$, $x_1=1+t$, $x_2=1$. Substitution of these values in (52) gives the quadratic of the section $x_0-x_1+x_2=0$ with C_1. From this, using quadratics $x_0=0$ and $x_2=0$ already found, the section $x_1=0$ is determined. The equations for C_1 thus found are

$$\left.\begin{aligned}x_0&=(\alpha_0\alpha_5-16\alpha_2\alpha_3)t^2-24\alpha_3^2t+6\alpha_2\alpha_5,\\ x_1&=24\alpha_2^2t^2+(52\alpha_2\alpha_3-\alpha_0\alpha_5)t+24\alpha_3^2,\\ x_2&=6\alpha_0\alpha_3t^2-24\alpha_2^2t+(\alpha_0\alpha_5-16\alpha_2\alpha_3).\end{aligned}\right\} \tag{60}$$

That these quadratics are not to be multiplied by factors independent of t is verified by substitution in (52).

The polar of the sixth double point (0, 1, 0) as to C_1 is $12(\alpha_2^2x_0+\alpha_3^2x_2)-(\alpha_0\alpha_5-16\alpha_2\alpha_3)x_1=0$, whence the quadratic of points of contact of tangents from this point is

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*If the general rational quintic is represented by $x_i=f_i(t)$, a line section ξ apolar to all line sections must satisfy $[f_i, \xi_0f_0+\xi_1f_1+\xi_2f_2]^5=0$. The conditions on ξ, namely $\xi_1(f_0, f_1)^5+\xi_2(f_0, f_2)^5=0$, $\xi_0(f_1, f_0)^5+\xi_2(f_1, f_2)^5=0$ and $\xi_0(f_2, f_0)^5+\xi_1(f_2, f_1)^5=0$ give $\xi_0:\xi_1:\xi_2=(f_1, f_2)^5:(f_2, f_0)^5:(f_0, f_1)^5$, which is unique when the f's are general.

$$t^2[12\alpha_2^2(\alpha_0\alpha_5-16\alpha_2\alpha_3)-6\cdot 12\alpha_0\alpha_3^3]$$
$$+t[(\alpha_0\alpha_5-15\alpha_2\alpha_3)(52\alpha_2\alpha_3-\alpha_0\alpha_5)+(24)^2\alpha_2^2\alpha_3^2]$$
$$+[12\alpha_3^2(\alpha_0\alpha_5-16\alpha_2\alpha_3)-6\cdot 12\alpha_2^3\alpha_5]=0.$$

The discriminant of this quadratic is the function whose vanishing is the condition that the six points lie on a conic.

To find the quintic of the five double points on C_1, we find the intersections of C_1 with any two adjoint cubics. The two sextics in t will have a quintic factor in common, which is the required quintic. This is found to be

$$F(t)=\alpha_0(\alpha_0\alpha_5+8\alpha_2\alpha_3)t^5+12(\alpha_0\alpha_3^2-8\alpha_2^3)t^4+16\alpha_2(\alpha_0\alpha_5-19\alpha_2\alpha_3)t^3$$
$$+16\alpha_3(\alpha_0\alpha_5-19\alpha_2\alpha_3)t^2+12(\alpha_5\alpha_2^2-8\alpha_3^3)t+\alpha_5(\alpha_0\alpha_5+8\alpha_2\alpha_3).$$

$F(t)$ is apolar to $f=\alpha_0t^5+10\alpha_1t^3+10\alpha_2t^2+\alpha_5$.

The six double points of O are completely determined as the fixed points of a certain (3, 1) correspondence. Let f be written $a_0t^5+a_1(1-t)^5-a_2$, and take N in the form $x_0=t^2$, $x_1=(1-t)^2$, $x_2=1$, as in § 1. Let t_1t_2 on N give point y, and let t_3, t_4, t_5 be roots of the cubic polar of t_1t_2 as to f. Lines of N at points t_3, t_4, t_5 form a triangle whose vertices are three points x corresponding to y. If t_3 and t_4 were given, we should find y as the intersection of the quadratic polars of these points as to f; hence, one x gives one y, and the correspondence is (3, 1). From the fact that a double point of O belongs to a quadratic which is a factor in its cubic polar, x and y coincide at the six double points and only there. The coordinates y must be rational functions of the coordinates x. Let x belong to the quadratic t_3t_4. The lines cutting out the quadratic polars (*i. e.*, the lines of K (§ 2) having these parameters) are

$$\begin{array}{lll} a_0t_3^3, & a_1(1-t_3)^3, & -a_2, \\ a_0t_4^3, & a_1(1-t_4)^3, & -a_2, \end{array}$$

whence

$$a_0y_0=(t_3+t_4)^2-t_3t_4+3[1-(t_3+t_4)],$$
$$a_1y_1=(t_3+t_4)^2-t_3t_4,$$
$$a_2y_2=(t_3+t_4)^2-t_3t_4+3t_3^2t_4^2-3t_3t_4(t_3+t_4).$$

Since $x_0=t_3t_4$, $x_1=(1-t_3)(1-t_4)$, $x_2=1$, and $x_0-x_1+x_2=t_3+t_4$, the transformation takes the form

$$a_0y_0=(-x_0+x_1+x_2)^2-x_1x_2,$$
$$a_1y_1=(x_0-x_1+x_2)^2-x_0x_2,$$
$$a_2y_2=(x_0+x_1-x_2)^2-x_0x_1.$$

The conics $y_i=0$ are on the point (1, 1, 1). Sets of three points x corresponding to one y are three variable intersections of this net.